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The Solution of Certain Integral Equations with Kernels $K(z,\zeta)/(z-\zeta)$

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1. Introduction.

It is well known that the solution of the integral equation

$$\oint_C \frac{\phi(z)}{z-\zeta}\, dz = h(\zeta)\phi(\zeta) + f(\zeta)$$

can be reduced to the Hilbert-Riemann boundary value problem which is defined in Section 2. This reduction, which depends on the ideas of Carleman and Plemelj, is explained in detail in the texts by Muskhelishvili [1], Gahov [2], and others; and it is now regarded as a standard method. It is the purpose of this paper to show that a more elementary method can be used to solve various types of integral equations involving Cauchy kernels provided certain conditions are satisfied. The function theoretic method described below avoids the analysis of a Hilbert-Riemann boundary value problem; consequently, when it can be used, it is more simple and rapid than the standard procedure. Furthermore, it can be used to solve some equations to which the standard method is inapplicable.

The method described in Section 3 involves the Plemelj formulas and little more than the theory of residues. It is of such an elementary nature that one would expect to find it in a place of some prominence in the texts, or surmise at least that it appears elsewhere. However, a more or less intensive search of the literature failed to reveal any reference to it. This led the author to believe that a presentation of it in this report might be useful.

Section 2 contains a review of the usual method for solving

$$\oint_C \frac{\phi(z)}{z - \zeta}\, dz = h(\zeta)\phi(\zeta) + f(\zeta) \quad .$$

In Section 3 an explicit solution is found for the more general integral equation

$$\oint_C \frac{K(z, \zeta)\phi(z)}{z - \zeta}\, dz = h(\zeta)\phi(\zeta) + f(\zeta) \quad .$$

Here, C is a closed curve and $h(\zeta)$ and $K(z, w)$ are required to satisfy certain analyticity conditions. Under these conditions, the introduction of a Hilbert-Riemann boundary value problem is not necessary. Section 4 is designed to indicate that the method of Section 3 can be extended and used to solve other types of functional equations whose kernels have singular parts which are simple poles.

2. The Standard Procedure.

In order to compare and contrast the standard method with the method given below, let us first see how the standard method is used to solve the equation

$$\oint_C \frac{\phi(z)}{z - \zeta}\, dz = h(\zeta)\phi(\zeta) + f(\zeta) \quad , \tag{2.1}$$

where C is a bounded, smooth, closed curve orientated in the counterclockwise direction. Since the form of the equation is not changed by a translation we may suppose without loss of generality that D^+, the interior of C, which

contains the positive side of C, also contains the origin. We assume that $f(\zeta)$ is such that if ζ and z lie on C then

$$|f(z) - f(\zeta)| \leq M|z - \zeta|^{\lambda}, \qquad 0 < \lambda \leq 1,$$

provided

$$|z - \zeta| < \delta .$$

We suppose that the quantities M, λ, and δ are constants. In other words, we assume that $f(\zeta)$ satisfies a uniform Hölder condition for ζ on C; and we require the unknown function $\phi(\zeta)$ to satisfy the same condition. This guarantees, in a simple way, the existence of the Cauchy principal value of the integral in (2.1). For the moment we assume that $h(\zeta)$ also satisfies a uniform Hölder condition, and that neither $h(\zeta) + \pi i$ nor $h(\zeta) - \pi i$ vanishes on C.

The idea of introducing the function

$$F(w) = \oint_C \frac{\phi(z)}{z - w}\, dz \tag{2.2}$$

to facilitate the analysis of (2.1) is due to T. Carleman. This function is sectionally holomorphic -- it is analytic for w in D^+, it is analytic for w in D^- -- and it vanishes like $1/w$ as w approaches infinity. The limit values of $F(w)$ as w approaches a point ζ on C in a nontangential direction from D^+ and D^- are respectively given by the Plemelj formulas

$$F^+(\zeta) = \pi i \phi(\zeta) + \oint_C \frac{\phi(z)}{z - \zeta}\, dz , \tag{2.3}$$

$$F^-(\zeta) = -\pi i \phi(\zeta) + \oint_C \frac{\phi(z)}{z - \zeta}\, dz . \tag{2.4}$$

The subtraction and addition of (2.3) and (2.4) yield

$$\phi(\zeta) = \frac{1}{2\pi i}[F^+(\zeta) - F^-(\zeta)] \quad , \tag{2.5}$$

$$\oint_C \frac{\phi(z)}{z-\zeta}\,dz = \frac{1}{2}[F^+(\zeta) + F^-(\zeta)] \tag{2.6}$$

and the substitution of these values in (2.1) gives

$$[h(\zeta) - \pi i]F^+(\zeta) - [h(\zeta) + \pi i]F^-(\zeta) = -2\pi i f(\zeta) \quad . \tag{2.7}$$

The problem of solving the equation (2.1) has now been reduced to a Hilbert-Riemann boundary value problem. This is the problem of finding a sectionally analytic function $F(w)$ which satisfies the barrier equation (2.7) and which possesses properties implied by the integral representation in (2.2). After such a function has been found the corresponding solution of (2.1) is given by (2.5).

If we assume that $F(w)$ can be factored in the form $F(w) = F_o(w)F_1(w)$, where $F_o(w)$ is an appropriate solution of the homogeneous equation

$$[h(\zeta) - \pi i]F_o^+(\zeta) - [h(\zeta) + \pi i]F_o^-(\zeta) = 0 \quad , \tag{2.8}$$

then $F_1(w)$ must satisfy

$$F_1^+(\zeta) - F_1^-(\zeta) = -\frac{2\pi i f(\zeta)}{[h(\zeta) - \pi i]F_o^+(\zeta)} \quad . \tag{2.9}$$

As can be seen from the Plemelj formulas, the general solution of (2.9) is

$$F_1(w) = -\oint_C \frac{f(z)}{[h(z) - \pi i]F_o^+(z)\cdot(z - w)}\, dz + p(w) \quad ,$$

where $p(w)$ must satisfy $p^+(\zeta) = p^-(\zeta)$.

Although we can proceed with any solution of (2.8) such that $1/F_o^+(z)$ is continuous on C, it is most convenient to seek a solution $F_o(w)$ which is analytic in D^+ and D^-. The homogeneous equation (2.8) can be written

$$\ln F_o^+(\zeta) - \ln F_o^-(\zeta) = \ln r(\zeta) \quad , \tag{2.10}$$

where

$$r(\zeta) = \frac{h(\zeta) + \pi i}{h(\zeta) - \pi i} \quad .$$

If $\ln r(\zeta)$ is Hölder continuous on C, an obvious solution of (2.10) is

$$\ln F_o(w) = \frac{1}{2\pi i} \oint_C \frac{\ln r(z)}{z - w}\, dz \quad .$$

We need, however, to provide for the fact that $\ln r(z)$ may change by a non-zero integral multiple m of $2\pi i$ as z traverses C. The integer m is the index of $r(z)$ on C and it is given by

$$m = \frac{1}{2\pi i} \oint_C \frac{r'(z)}{r(z)}\, dz \quad .$$

Let us take

$$F_o(w) = \begin{cases} G_o(w) \quad , & w \text{ in } D^+ \quad , \\ \dfrac{G_o(w)}{w^m} \quad , & w \text{ in } D^- \quad . \end{cases}$$

With this, equation (2.10) becomes

$$\ln G_o^+(\zeta) - \ln G_o^-(\zeta) = \ln \frac{r(\zeta)}{\zeta^m} \tag{2.11}$$

and this is satisfied by

$$G_o(w) = \exp \frac{1}{2\pi i} \oint_C \ln \left[\frac{r(z)}{z^m}\right] \frac{1}{z-w}\, dz .$$

We now have

$$F(w) = -F_o(w) \oint_C \frac{f(z)}{[h(z) - \pi i] F_o^+(z)(z-w)}\, dz + F_o(w) p(w) , \tag{2.12}$$

where $p(w)$ must be an entire function. Since $F(w)$ must behave like $1/w$ as $w \to \infty$ and since $F_o(w) \to 1/w^m$ as $w \to \infty$ it follows that $p(w)$ can be no more than a polynomial $p_{m-1}(w)$ of degree $m-1$ if $m > 0$. If $m \leq 0$, $p(w) \equiv 0$. If $m < 0$, $F(w)$ exists only if

$$\int_{w \to \infty} F_o(w) \oint_C \frac{f(z)}{[h(z) - \pi i] F_o^+(z)(z-w)}\, dz = 0 ,$$

that is, only if

$$\oint_C \frac{z^k f(z)}{[h(z) - \pi i] F_o^+(z)}\, dz = 0 , \qquad k = 0, 1, \ldots, -(m+1) . \tag{2.13}$$

The substitution of (2.12) in (2.5) gives the solution of (2.1). It is

$$\phi(\zeta) = -\frac{h(\zeta) f(\zeta)}{h^2(\zeta) + \pi^2} - \frac{G_o^+(\zeta)}{h(\zeta) + \pi i} \left[\oint_C \frac{f(z)}{[h(z) - \pi i] G_o^+(z)(z-\zeta)}\, dz - p_{m-1}(\zeta)\right]. \tag{2.14}$$

The analysis we have just presented follows that given in Muskhelishvili's book [1].

Let us assume now that each of $h(\zeta) + \pi i$ and $h(\zeta) - \pi i$ is analytic in $D^+ + C$ with simple zeros in D^+ respectively at

$$\alpha_k \;, \qquad k = 1, 2, \ldots, n_1 \;,$$

$$\beta_k \;, \qquad k = 1, 2, \ldots, n_2 \;.$$

For this case, the index of

$$r(\zeta) = \frac{h(\zeta) + \pi i}{h(\zeta) - \pi i}$$

is $m = n_1 - n_2$. With these assumptions the function $G_o^+(\zeta)$ is

$$G_o^+(\zeta) = \exp\left\{ \ln \frac{r(\zeta)}{\zeta^m} + \frac{1}{2\pi i} \oint_{C_1} \ln \left[\frac{r(z)}{z^m}\right] \frac{1}{z - \zeta} \, dz \right\} ,$$

where C_1 is a path in D^+ containing the zeros and infinities of $r(z)/z^m$. An integration by parts gives

$$G_o^+(\zeta) = \exp \left\{ \begin{array}{l} \ln \dfrac{1}{\zeta^m} \left[\dfrac{h(\zeta) + \pi i}{h(\zeta) - \pi i}\right] \\ \qquad - \dfrac{1}{2\pi i} \displaystyle\oint_{C_1} \left[- \frac{m}{z} + \frac{h'(z)}{h(z) + \pi i} - \frac{h'(z)}{h(z) - \pi i} \right] \ln(z - \zeta) dz \end{array} \right\}$$

and the theory of residues shows

$$G_o^+(\xi) = \left[\frac{h(\xi) + \pi i}{h(\xi) - \pi i}\right] \frac{\prod_{k=1}^{n_2} (\xi - \beta_k)}{\prod_{k=1}^{n_1} (\xi - \alpha_k)} .$$

Hence the solution of the integral equation (2.1) is

$$\phi(\xi) = - \frac{h(\xi) f(\xi)}{h^2(\xi) + \pi^2} - \frac{1}{[h(\xi) - \pi i]} \cdot \frac{\prod (\xi - \beta_k)}{\prod (\xi - \alpha_k)} \cdot$$

(2.15)

$$\cdot \left[\oint_C \frac{\prod (z - \alpha_k)}{\prod (z - \beta_k)} \cdot \frac{f(z)}{[h(z) + \pi i](z - \xi)} dz - p_{n_1 - n_2 - 1}(\xi) \right]$$

provided $h(\xi)$ satisfies the analyticity assumption stated above.

After some manipulation and use of (2.13) when $n_2 \geq n_1$, equation (2.15) can be brought to the form

$$\phi(\xi) = - \frac{h(\xi) f(\xi)}{h^2(\xi) + \pi^2} - \frac{1}{h(\xi) - \pi i} \oint_C \frac{f(z)}{[h(z) + \pi i](z - \xi)} dz$$

(2.16)

$$- \frac{P_{n_1 - 1}(\xi)}{[h(\xi) - \pi i] \prod_{k=1}^{n_1} (\xi - \alpha_k)} ,$$

where $P_{n_1 -1}(\xi)$ is a polynomial of degree $n_1 - 1$. The arbitrariness of the solution implied by $P_{n_1 -1}(\xi)$ can only be removed by imposing side conditions on $\phi(\xi)$.

The solution (2.16) of (2.1) can be obtained more easily without the introduction of a Hilbert-Riemann problem if we use the following technique which may be new. Introduce

(2.17) $$F(w) = \oint_C \frac{\phi(z)}{z - w}\, dz$$

and hence

(2.18) $$F^+(\zeta) = \pi i \phi(\zeta) + \oint_C \frac{\phi(z)}{z - \zeta}\, dz \quad .$$

If $\phi(z)$ is to satisfy (2.1) we must have

(2.19) $$F^+(\zeta) = [\pi i + h(\zeta)]\phi(\zeta) + f(\zeta)$$

or

(2.20) $$\phi(\zeta) = \frac{F^+(\zeta) - f(\zeta)}{h(\zeta) + \pi i} \quad .$$

The substitution of this in (2.1) gives

(2.21) $$\oint_C \frac{[F^+(z) - f(z)]}{[h(z) + \pi i](z - \zeta)}\, dz = h(\zeta)\phi(\zeta) + f(\zeta) \quad .$$

In accordance with our assumption the zeros of $h(\zeta) + \pi i$ lie in a domain D_1^+ which, with its counterclockwise orientated boundary C_1, is contained in D^+. In the integral involving $F^+(z)$ in (2.21) let C be contracted into C_1. Since $F(w)$ is analytic in D^+, equation (2.21) gives

$$h(\zeta)\phi(\zeta) + f(\zeta) = \frac{\pi i F^+(\zeta)}{h(\zeta) + \pi i} - \oint_C \frac{f(z)\, dz}{[h(z) + \pi i](z - \zeta)} + \oint_{C_1} \frac{F(z)\, dz}{[h(z) + \pi i](z - \zeta)}$$

or, using (2.19),

$$h(\zeta)\phi(\zeta) + f(\zeta) = \pi i\phi(\zeta) + \frac{\pi i f(\zeta)}{h(\zeta) + \pi i} - \oint_C \frac{f(z)\,dz}{[h(z) + \pi i](z - \zeta)}$$

$$+ \oint_{C_1} \frac{F(z)\,dz}{[h(z) + \pi i](z - \zeta)}$$

from which we find

$$\phi(\zeta) = -\frac{h(\zeta)f(\zeta)}{h^2(\zeta) + \pi^2} - \frac{1}{[h(\zeta) - \pi i]} \oint_C \frac{f(z)\,dz}{[h(z) + \pi i](z - \zeta)}$$

(2.22)

$$+ \frac{1}{[h(\zeta) - \pi i]} \oint_{C_1} \frac{F(z)\,dz}{[h(z) + \pi i](z - \zeta)} .$$

If $h(z) + \pi i$ is analytic in $D^+ + C$ with simple zeros in D^+ at α_k, $k = 1, 2, \ldots, n_1$, equation (2.22) is the same as

$$\phi(\zeta) = -\frac{h(\zeta)f(\zeta)}{h^2(\zeta) + \pi^2} - \frac{1}{[h(\zeta) - \pi i]} \oint_C \frac{f(z)\,dz}{[\alpha(z) + \pi i](z - \zeta)}$$

(2.23)

$$+ \frac{1}{[h(\zeta) - \pi i]} \sum_{k=1}^{n_1} \frac{c_k}{\zeta - \alpha_k}$$

which agrees with (2.16). Thus we see that the method which has just been used leads quickly to a form of the solution of (2.1). The analysis, however, is not complete until we find the special values of the constants c_k, and the conditions on $f(z)$, that are required for (2.23) to actually satisfy (2.1). We return to this point below in connection with a more general equation.

3. The Kernel $K(z,\zeta)/(z-\zeta)$.

Let us consider the equation

$$(3.1) \qquad \oint_C \frac{K(z,\zeta)\phi(z)\,dz}{z-\zeta} = h(\zeta)\phi(\zeta) + f(\zeta) \quad ,$$

where C, $\phi(\zeta)$ and $f(\zeta)$ are required to satisfy the conditions imposed for equation (2.1). Most of the extensive work on equation (3.1) is based on the determination and study of equivalent Fredholm equations. It has also been explicitly solved for kernels $K(z,\zeta)/(z-\zeta)$ of such character that (3.1) is reducible to a barrier equation to be satisfied along C plus other paths. See Gahov [2]. We proceed to show that a simple function theoretic method can be used to find the explicit solution of (3.1) provided the following conditions are met:

1. $K(z, w)$ is an analytic function of either variable when each of z and w is in $D^+ + C$.

2. $h(\zeta)$ is analytic in $D^+ + C$.

3. Neither $h(\zeta) + \pi i K(\zeta,\zeta)$ nor $h(\zeta) - \pi i K(\zeta,\zeta)$ vanishes on C.

4. $K(\zeta,\zeta) \neq 0$ for ζ on C.

These conditions do not necessarily imply that (3.1) can be reduced to a Hilbert-Riemann problem.

The function

$$(3.2) \qquad F(w) = \oint_C \frac{K(z,w)\phi(z)}{z-w}\,dz$$

is analytic for w in D^+ and we have

$$F^+(\zeta) = \pi i K(\zeta, \zeta)\phi(\zeta) + \oint_C \frac{K(z, \zeta)\phi(z)}{z - \zeta}\, dz \quad . \tag{3.3}$$

Since (3.1) is to be satisfied, this limit value is

$$F^+(\zeta) = [h(\zeta) + \pi i K(\zeta, \zeta)]\phi(\zeta) + f(\zeta) \tag{3.4}$$

from which

$$\phi(\zeta) = \frac{F^+(\zeta) - f(\zeta)}{[h(\zeta) + \pi i K(\zeta, \zeta)]} \quad . \tag{3.5}$$

The substitution of this in (3.1) gives

$$\oint_C \frac{F^+(z)K(z, \zeta)\, dz}{[h(z) + \pi i K(z, z)](z - \zeta)} - \oint_C \frac{f(z)K(z, \zeta)\, dz}{[h(z) + \pi i K(z, z)](z - \zeta)} \tag{3.6}$$

$$= h(\zeta)\phi(\zeta) + f(\zeta) \quad .$$

Let C_o be a counterclockwise path in D^+ and let it be the smooth boundary of a domain D_o^+, in D^+, which contains not only the zeros of $h(z) + \pi i K(z, z)$, but also for reasons evident later, the zeros of $h(z) - \pi i K(z, z)$. The deformation of C in the first integral of (3.6) into C_o produces

$$h(\zeta)\phi(\zeta) + f(\zeta) = \frac{\pi i F^+(\zeta)K(\zeta, \zeta)}{h(\zeta) + \pi i K(\zeta, \zeta)} - \oint_C \frac{f(z)K(z, \zeta)\, dz}{[h(z) + \pi i K(z, z)](z - \zeta)}$$

$$+ \oint_{C_o} \frac{F(z)K(z, \zeta)\, dz}{[h(z) + \pi i K(z, z)](z - \zeta)}$$

or, from (3.5)

$$h(\zeta)\phi(\zeta) + f(\zeta) = \pi i K(\zeta,\zeta)\phi(\zeta) + \frac{\pi i K(\zeta,\zeta) f(\zeta)}{h(\zeta)+\pi i K(\zeta,\zeta)}$$

$$- \oint_C \frac{f(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)} + \oint_{C_o} \frac{F(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)} .$$

Hence we see that $\phi(\zeta)$ can be expressed as

$$\phi(\zeta) = - \frac{h(\zeta)f(\zeta)}{h^2(\zeta)+\pi^2 K^2(\zeta,\zeta)} - \frac{1}{[h(\zeta)-\pi i K(\zeta,\zeta)]} \oint_C \frac{f(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)}$$

(3.7)

$$+ \frac{1}{[h(\zeta)-\pi i K(\zeta,\zeta)]} \oint_{C_o} \frac{F(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)} .$$

Since $F(z)$ is analytic in D^+, the integral along C_o can be calculated in terms of prescribed functions. If $h(z) + \pi i K(z,z)$ possesses no zeros in D^+ then the integral along C_o is zero. If the zeros of $h(z) + \pi i K(z,z)$ in D^+ are situated at α_k, $k = 1, 2, \ldots, n_1$, and if the multiplicity of α_k is m_k we can write

$$\oint_{C_o} \frac{F(z)K(z,\zeta)\,dz}{[h(z) + \pi i K(z,z)](z-\zeta)} = \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} H_{kj}(\zeta) ,$$

where $\sum_{j=1}^{m_k} c_{kj} H_{kj}(\zeta)$ is $2\pi i$ times the residue of $F(z)K(z,\zeta)/[h(z)+\pi i K(z,z)](z-\zeta)$ at $z = \alpha_k$. In fact, if $G(z)$ is a polynomial of degree $(\sum_{k=1}^{n_1} m_k) - 1$, the supposition

$$\frac{F(z)}{h(z)+\pi i K(z,z)} = \frac{G(z)}{\prod_{k=1}^{n_1} (z-\alpha_k)^{m_k}}$$

leads to no loss of generality; then, using the partial fraction expansion,

$$\frac{G(z)}{\prod_{k=1}^{n_1} (z - \alpha_k)^{m_k}} = \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} \frac{c_{kj}}{(z - \alpha_k)^j}$$

we can write

$$\oint_{C_o} \frac{F(z)\, K(z, \zeta)\, dz}{[h(z) + \pi i\, K(z, z)](z - \zeta)} = \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(z, \zeta)\, dz}{(z - \zeta)(z - \alpha_k)^j}$$

$$= \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \frac{2\pi i}{(j-1)!} \frac{\partial^{j-1}}{\partial z^{j-1}} \left[\frac{K(z, \zeta)}{z - \zeta} \right] \Bigg|_{z = \alpha_k} ,$$

where γ_k is a circle centered at α_k and containing no other zero of $h(z) + \pi i\, K(z, z)$. It is also possible to express c_{kj} in terms of integrals involving $\phi(z)$ because we can write

$$\oint_{C_o} \frac{F(z)\, K(z, \zeta)\, dz}{[h(z) + \pi i\, K(z, z)](z - \zeta)} = \oint_{C_o} \frac{K(z, \zeta)}{[h(z) + \pi i\, K(z, z)](z - \zeta)} \oint_{C} \frac{K(t, z)\, \phi(t)\, dt\, dz}{t - z}$$

$$= \oint_{C} \phi(t) \oint_{C_o} \frac{K(z, \zeta)\, K(t, z)}{[h(z) + \pi i\, K(z, z)](z - \zeta)(t - z)} \, dz\, dt .$$

Thus we conclude that if a solution of (3.1) exists it must be possible to express it in the form

$$\phi(\zeta) = -\frac{h(\zeta)f(\zeta)}{h^2(\zeta)+\pi^2 K(\zeta,\zeta)} - \frac{1}{[h(\zeta)-\pi i K(\zeta,\zeta)]} \oint_C \frac{f(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)}$$

(3.8)

$$+ \frac{1}{[h(\zeta)-\pi i K(\zeta,\zeta)]} \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(z,\zeta)\,dz}{(z-\zeta)(z-\alpha_k)^j} \quad .$$

On the other hand, a solution of (3.1) may not actually exist unless $f(z)$ and/or the constants c_{kj} are subject to more conditions than we have so far admitted. In order to investigate this, let us substitute (3.3) into (3.1) and use the Hardy-Poincaré-Bertrand formula. This formula has to do with the interchange of order of integration in the double integral of a function of the type $H(z,\zeta,\sigma)/(\zeta-\sigma)(z-\zeta)$ which has simple poles. In a form sufficient for our purposes, the formula asserts that

$$\oint_C \frac{H_1(z,\zeta)}{z-\zeta} \oint_C \frac{H_2(t,z)}{t-z}\,dt\,dz = -\pi^2 H_1(\zeta,\zeta)H_2(\zeta,\zeta)$$

$$+ \oint_C \oint_C \frac{H_1(z,\zeta)H_2(t,z)}{(z-\zeta)(t-z)}\,dz\,dt$$

provided $H_1(z,\zeta)$, $H_2(t,z)$ satisfy uniform Hölder conditions with respect to each variable. As a partial fraction expansion shows we can write the last equation as

$$\oint_C \frac{H_1(z,\zeta)}{z-\zeta} \oint_C \frac{H_2(t,z)}{t-z}\,dt\,dz = -\pi^2 H_1(\zeta,\zeta)H_2(\zeta,\zeta)$$

$$+ \oint_C \frac{1}{t-\zeta} \oint_C H_1(z,\zeta)H_2(t,z)\left[\frac{1}{z-\zeta} - \frac{1}{z-t}\right] dz\,dt \quad .$$

The result of using this formula after substituting (3. 8) in (3. 1) is

$$- \oint_C \frac{f(t)}{[h(t)+\pi i K(t,t)]} \cdot \frac{1}{(t-\zeta)} \oint_{C_o} \frac{K(t,z)K(z,\zeta)}{[h(z)-\pi i K(z,z)]} \left[\frac{1}{z-\zeta} - \frac{1}{z-t} \right] dz\, dt$$

$$+ \oint_C \frac{K(z,\zeta)}{[h(z)-\pi i K(z,z)]} \cdot \frac{1}{(z-\zeta)} \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t,z)}{(t-z)(t-\alpha_k)^j} dt\, dz$$

$$= \frac{h(\zeta)}{[h(\zeta)-\pi i K(\zeta,\zeta)]} \cdot \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t,\zeta)\, dt}{(t-\zeta)(t-\alpha_k)^j}$$

and, after deforming the path C in the second term above,

$$- \oint_C \frac{f(t)}{[h(t)+\pi i K(t,t)]} \cdot \frac{1}{(t-\zeta)} \oint_{C_o} \frac{K(t,z)K(z,\zeta)}{[h(z)-\pi i K(z,z)]} \left[\frac{1}{z-\zeta} - \frac{1}{z-t} \right] dz\, dt$$

(3. 9)
$$+ \oint_{C_o} \frac{K(z,\zeta)}{[h(z)-\pi i K(z,z)]} \cdot \frac{1}{(z-\zeta)} \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t,z)}{(t-z)(t-\alpha_k)^j} dt\, dz$$

$$= \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t,\zeta)}{(t-\zeta)(t-\alpha_k)^j} dt \quad .$$

Since the satisfaction of (3. 9) proves the existence of the solution (3. 8) of (3. 1), we will refer to (3. 9) or one of its equivalent forms as the existence condition.

The existence condition (3.9) can be expressed in different ways. Let us assume hereafter that $K(z, z) \neq 0$ for z in $D^{+} + C$. The sum on the right of (3.9) can then be written

$$\sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t, \zeta)}{(t-\zeta)(t-\alpha_k)^j} \, dt$$

$$= -\frac{1}{2\pi i} \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{C_o} \frac{K(z, \zeta)}{K(z, z)} \frac{1}{(z-\zeta)} \oint_{\gamma_k} \frac{K(t, z)}{(t-\alpha_k)^j (t-z)} \, dt \, dz$$

$$= -\frac{1}{2\pi i} \oint_{C_o} \frac{K(z, \zeta)}{K(z, z)} \frac{1}{z-\zeta} \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t, z)}{(t-z)(t-\alpha_k)^j} \, dt \, dz$$

as an interchange of order of integration shows, and consequently we can combine the last integral with the second integral of (3.9) to get

$$-\oint_{C} \frac{f(t)}{[h(t)+\pi i K(t, t)]} \cdot \frac{1}{(t-\zeta)} \cdot \oint_{C_o} \frac{K(t, z) K(z, \zeta)}{[h(z)-\pi i K(z, z)]} \cdot \left[\frac{1}{z-\zeta} - \frac{1}{z-t}\right] dz \, dt$$

$$+ \frac{1}{2\pi i} \oint_{C_o} \frac{K(z, \zeta)}{(z-\zeta) K(z, z)} \left[\frac{h(z)+\pi i K(z, z)}{h(z)-\pi i K(z, z)}\right] \cdot \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t, z) \, dt \, dz}{(t-z)(t-\alpha_k)^j}$$

$$= 0$$

or

$$\frac{1}{2\pi i} \oint_{C_o} \frac{K(z,\zeta)}{(z-\zeta)K(z,z)} \cdot \left[\frac{h(z)+\pi i K(z,z)}{h(z)-\pi i K(z,z)}\right] \cdot \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t,z)\, dt\, dz}{(t-z)(t-\alpha_k)^j}$$

(3.10)

$$= \oint_{C_o} \frac{K(z,\zeta)}{(z-\zeta)[h(z)-\pi i K(z,z)]} \cdot \oint_C \frac{f(t)K(t,z)}{[h(t)+\pi i K(t,t)](t-z)}\, dt\, dz$$

which is the condition for the existence of a solution of (3.1) when $K(z,z) \neq 0$ for z in $D^+ + C$. Note that

$$[h(z) + \pi i K(z,z)] \cdot \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} c_{kj} \oint_{\gamma_k} \frac{K(t,z)}{(t-z)(t-\alpha_k)^j}\, dt$$

is an analytic function for z on C_o or in its interior, and so also is

$$\oint_C \frac{f(t)K(t,z)}{[h(t)+\pi i K(t,t)](t-z)}\, dt$$

since z in this integral as it appears in (3.10) is in the interior of C. The integrals in (3.10) can therefore be easily evaluated in terms of the residues at the zeros of $h(z) - \pi i K(z,z)$ and $h(z) + \pi i K(z,z)$.

Suppose next that $h(z) - \pi i K(z,z)$ does not possess any zeros in $D^+ + C$. Then (3.10) is automatically satisfied and we have proved that the solution of

$$\oint_C \frac{K(z,\zeta)\phi(z)}{z-\zeta}\, dz = h(\zeta)\phi(\zeta) + f(\zeta)$$

is

$$\phi(\zeta) = -\frac{h(\zeta)f(\zeta)}{h^2(\zeta)+\pi^2K^2(\zeta,\zeta)} - \frac{1}{[h(\zeta)-\pi i K(\zeta,\zeta)]}\oint_C \frac{K(z,\zeta)f(z)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)}$$

(3.11)

$$+ \frac{1}{[h(\zeta)-\pi i K(\zeta,\zeta)]}\oint_{C_o} \frac{F(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)} \quad ,$$

where

$$h(z) - \pi i K(z,z) \neq 0 \quad , \qquad K(z,z) \neq 0$$

for z in $D^+ + C$ and $F(z)$ is an arbitrary analytic function for z in $D^+ + C$. The formula (3.11) subsumes some well-known results, and others, which if they have ever been noted, are not so well known. For example, if $K(z,\zeta) \equiv 1$, and $h(\zeta) = \lambda$, (3.11) shows that

(3.12)
$$\oint_C \frac{\phi(z)}{z-\zeta}\,dz = \lambda\phi(\zeta) + f(\zeta)$$

is satisfied by the well-known solution

(3.13)
$$\phi(\zeta) = -\frac{\lambda f(\zeta)}{\lambda^2+\pi^2} - \frac{1}{\lambda^2+\pi^2}\oint_C \frac{f(z)}{z-\zeta}\,dz$$

provided $\lambda \neq \pm\pi i$. Again, if, for example, $K(z,\zeta) = K(z-\zeta)$ with $K(0) = 1$, and $h(z) - \pi i$ does not vanish in $D^+ + C$, (3.11) shows that the solution of

(3.14)
$$\oint_C \frac{K(z-\zeta)\phi(z)}{z-\zeta}\,dz = h(\zeta)\phi(\zeta) + f(\zeta)$$

is

$$\phi(\zeta) = -\frac{h(\zeta)f(\zeta)}{h^2(\zeta)+\pi^2} - \frac{1}{[h(\zeta)-\pi i]}\oint_C \frac{K(z-\zeta)f(z)}{[h(z)+\pi i](z-\zeta)}\,dz$$

(3.15)

$$+ \frac{1}{[h(\zeta)-\pi i]}\oint_{C_o} \frac{F(z)K(z-\zeta)}{[h(z)+\pi i](z-\zeta)}\,dz \quad .$$

As a particular case of (3.14) we see that the solution of

(3.16) $$\oint_C \frac{K(z-\zeta)\phi(z)}{z-\zeta}\,dz = \lambda\phi(\zeta) + f(\zeta)$$

is

(3.17) $$\phi(\zeta) = -\frac{\lambda f(\zeta)}{\lambda^2+\pi^2} - \frac{1}{\lambda^2+\pi^2}\oint_C \frac{K(z-\zeta)f(z)}{z-\zeta}\,dz$$

provided $\lambda \neq \pm \pi i$. As another example, if $K(z, z) \neq 0$ for z in $D^+ + C$, the solution of

(3.18) $$\oint_C \frac{K(z,\zeta)\phi(z)}{z-\zeta}\,dz = f(\zeta)$$

is

(3.19) $$\phi(\zeta) = -\frac{1}{\pi^2 K(\zeta,\zeta)}\oint_C \frac{K(z,\zeta)f(z)}{K(z,z)(z-\zeta)}\,dz \quad .$$

Many other special cases of (3.11) could be noted, and we could of course also examine some of the equations which result when we specialize C -- regard it, say, as a circle. We refrain, however, from further particularizations at this time because it is more interesting to go back to (3.10).

Let us return to a consideration of (3.10) and suppose that $h(z) - \pi i K(z,z)$ does possess zeros for z in D^+. For simplicity, take the case in which $h(z) - \pi i K(z,z)$ possesses n_2 simple zeros in D^+ at $z = \beta_\ell$, $\ell = 1, 2, \ldots, n_2$. No one of these zeros can coincide with α_k, a zero of $h(z) + \pi i K(z,z)$, because if this were the case we would have

$$\left.\begin{aligned} h(\beta_\ell) &= \pi i K(\beta_\ell, \beta_\ell) \quad , \\ h(\alpha_k) &= -\pi i K(\alpha_k, \alpha_k) \end{aligned}\right. \qquad \beta_\ell = \alpha_k$$

from which, by subtraction, $K(\beta_\ell, \beta_\ell) = 0$ which contradicts our assumption $K(z,z) \neq 0$, z in D^+. The circles γ_k in (3.10) can therefore be taken so small that each contains no β_ℓ. Let us also assume, for simplicity, that the zeros of $h(z) + \pi i K(z,z)$ are simple, i.e., $m_k = 1$ in (3.10), and let us write $c_{k1} = c_k$. Then the evaluation of the integrals in the existence condition (3.10) gives

$$\sum_{\ell=1}^{n_2} \frac{K(\beta_\ell, \zeta)}{(\beta_\ell - \zeta) K(\beta_\ell, \beta_\ell)} \cdot \frac{2\pi i K(\beta_\ell, \beta_\ell)}{[h'(\beta_\ell) - \pi i K'(\beta_\ell, \beta_\ell)]} \cdot \sum_{k=1}^{n_1} c_k \oint_{\gamma_k} \frac{K(t, \beta_\ell)}{(t-\beta_\ell)(t-\alpha_k)}\, dt$$

$$= 2\pi i \sum_{\ell=1}^{n_2} \frac{K(\beta_\ell, \zeta)}{(\beta_\ell - \zeta)[h'(\beta_\ell) - \pi i K'(\beta_\ell, \beta_\ell)]} \cdot \oint_C \frac{f(t) K(t, \beta_\ell)\, dt}{[h(t) + \pi i K(t,t)](t - \beta_\ell)} \ .$$

This implies that if a solution is to exist, the constants c_k and the prescribed function $f(t)$ must satisfy

$$(3.20) \qquad \sum_{k=1}^{n_1} \frac{c_k K(\alpha_k, \beta_\ell)}{\alpha_k - \beta_\ell} = \oint_C \frac{f(t) K(t, \beta_\ell)\, dt}{[h(t) + \pi i K(t,t)](t - \beta_\ell)} \ , \qquad \ell = 1, 2, \ldots, n_2.$$

This is the existence condition when the zeros of $h(\zeta) - \pi i K(\zeta, \zeta)$ and $h(\zeta) + \pi i K(\zeta, \zeta)$ are simple.

We have here in (3. 20) a system of n_2 linear algebraic equations for the determination of n_1 constants. The analysis of a system of this type in terms of the rank of the matrix of the coefficients of the unknowns, and the augmented matrix, is so well known that it need not be repeated here. We remark, however, that if $f(t)$ is specified, then such an analysis leads to one of three conclusions:

1. The system is inconsistent. In other words, it is not possible to find a set of constants $\{c_k\}$ which satisfies (3. 20). In this case, a solution of (3. 1) does not exist.

2. The system (3. 20) is satisfied by just one set of values for $\{c_k\}$. In this case, (3. 1) possesses a unique solution.

3. The constants c_k are linearly dependent on $r < n_1$ of their number which may be assigned arbitrary values. In this case, the solution of (3. 1) [as we see from (3. 3)] is given by

$$\phi(\zeta) = - \frac{h(\zeta) f(\zeta)}{h^2(\zeta) + \pi^2 K(\zeta, \zeta)} - \frac{1}{[h(\zeta) - \pi i K(\zeta, \zeta)]} \oint_C \frac{f(z) K(z, \zeta)\, dz}{[h(z) + \pi i K(z, z)](z - \zeta)}$$

$$+ \frac{2\pi i}{[h(\zeta) - \pi i K(\zeta, \zeta)]} \cdot \sum_{k=1}^{n_1} \frac{c_k K(\alpha_k, \zeta)}{\alpha_k - \zeta}$$

which contains r arbitrary constants, with the remaining constants to be determined by (3. 20). The arbitrary constants can be determined numerically by imposing side conditions on $\phi(\zeta)$.

We also note that the analysis of (3. 20) can be related to the solutions of the adjoint homogeneous equation associated with (3. 1) namely

$$(3.21) \qquad \oint_C \frac{K(\zeta, z)\psi(z)\,dz}{z - \zeta} = -\,h(\zeta)\psi(\zeta) \quad .$$

Under the assumption that the zeros of $h(\psi) + \pi i K(z, z)$ and $h(z) - \pi i K(z, z)$ in D^+ are simple, the solution of (3. 21) is

$$(3.22) \qquad \psi(\zeta) = \frac{1}{[h(\zeta) + \pi i K(\zeta, \zeta)]} \cdot \sum_{\ell=1}^{n_2} a_\ell \frac{K(\zeta, \beta_\ell)}{\zeta - \beta_\ell}$$

which can be read off from (3. 8). We deduce from (3. 10) [after we replace $K(z, \zeta)$ with $K(\zeta, z)$ and $h(z)$ with $-\,h(z)$] that the constants in (3. 22) must satisfy

$$(3.23) \qquad \sum_{\ell=1}^{n_2} a_\ell \frac{K(\alpha_k, \beta_\ell)}{\alpha_k - \beta_\ell} = 0 \quad , \qquad\qquad k = 1, 2, \ldots, n_1 \quad .$$

Now if we multiply the ℓ-th equation of (3. 20) by a_ℓ and add the equations we have

$$\sum_{k=1}^{n_1} c_k \sum_{\ell=1}^{n_2} \frac{a_\ell K(\alpha_k, \beta_\ell)}{\alpha_k - \beta_\ell} = \oint_C \frac{f(t)}{[h(t) + \pi i K(t, t)]} \cdot \sum_{\ell=1}^{n_2} \frac{a_\ell K(t, \beta_\ell)}{t - \beta_\ell} \cdot dt$$

and hence from (3. 23) and (3. 22)

$$\oint_C f(t)\psi(t)\,dt = 0 \quad .$$

In other words, a necessary condition for the existence of a solution of (3.1) is that $f(t)$ be orthogonal to each solution of (3.21). It can easily be proved that this condition is also a sufficient condition.

The solution of

$$\oint_C \frac{K(z,\zeta)\phi(z)\,dz}{z-\zeta} = h(\zeta)\phi(\zeta) + f(\zeta) \tag{3.24}$$

is more simple than what we have above if $f(z)$ is analytic in $D^+ + C$. Under this assumption, with the analyticity conditions already imposed on $h(z)$ and $K(z,\zeta)$, we can deform the path C in (3.7) into C_o. This gives

$$\phi(\zeta) = -\frac{f(\zeta)}{h(\zeta)-\pi i K(\zeta,\zeta)} + \frac{1}{[h(\zeta)-\pi i K(\zeta,\zeta)]}\oint_{C_o} \frac{F_1(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)}, \tag{3.25}$$

where $F_1(z) = F(z) - f(z)$ is analytic in D^+. The solution (3.25) can be written

$$\begin{aligned}\phi(\zeta) &= -\frac{f(\zeta)}{[h(\zeta)-\pi i K(\zeta,\zeta)]}\\ &+ \frac{1}{[h(\zeta)-\pi i K(\zeta,\zeta)]}\cdot\sum_{k=1}^{n_1}\sum_{j=1}^{m_k} b_{kj}\oint_{\gamma_k}\frac{K(z,\zeta)\,dz}{(z-\zeta)(z-\alpha_k)^j}\end{aligned} \tag{3.26}$$

and, if $K(z,z) \neq 0$, the existence condition is

$$\frac{1}{2\pi i} \oint_{C_o} \frac{K(z,\zeta)}{(z-\zeta)K(z,z)} \cdot \left[\frac{h(z)+\pi i K(z,z)}{h(z)-\pi i K(z,z)}\right]$$

(3.27)

$$\cdot \sum_{k=1}^{n_1} \sum_{j=1}^{m_k} b_{kj} \oint_{\gamma_k} \frac{K(t,z)\,dt\,dz}{(t-z)(t-\alpha_k)^j} = \oint_{C_o} \frac{f(z)K(z,\zeta)\,dz}{[h(z)-\pi i K(z,z)](z-\zeta)} .$$

We conclude this section with a brief summary of the technique we have used. Subject to the stated analyticity conditions on $K(z,\zeta)$ and $h(\zeta)$ we have seen that the equation

(3.28)
$$\oint_C \frac{K(z,\zeta)\phi(z)\,dz}{z-\zeta} = h(\zeta)\phi(\zeta) + f(\zeta)$$

can be written

$$[h(\zeta)+\pi i K(\zeta,\zeta)]\phi(\zeta) + f(\zeta) = \oint_C \frac{K(z,\zeta)\phi(z)\,dz}{z-\zeta} + \pi i \phi(\zeta) K(\zeta,\zeta)$$

(3.29)

$$= F^+(\zeta) ,$$

where $F^+(\zeta)$ is the Hölder continuous limit value on C of a function $F(z)$ analytic in the interior of C. If we solve (3.29) for $\phi(\zeta)$ and substitute in the integral of (3.28) we have

$$h(\zeta)\phi(\zeta) + f(\zeta) = \oint_C \frac{F^+(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)}$$

(3.30)

$$- \oint_C \frac{f(z)K(z,\zeta)\,dz}{[h(z)+\pi i K(z,z)](z-\zeta)} .$$

The deformation of the path C in the first integral of (3. 30) into a path C_o in D^+ which contains in its interior the zeros of $h(z) + \pi i K(z, z)$ and $h(z) - \pi i K(z, z)$ shows that if a solution of (3. 28) exists it must be

$$\phi(\zeta) = - \frac{h(\zeta) f(\zeta)}{h^2(\zeta)+\pi^2 K^2(\zeta, \zeta)} - \frac{1}{[h(\zeta)-\pi i K(\zeta, \zeta)]} \cdot \oint_C \frac{f(z) K(z, \zeta)\, dz}{[h(z)+\pi i K(z, z)](z-\zeta)}$$

(3. 31)

$$+ \frac{1}{[h(\zeta)-\pi i K(\zeta, \zeta)]} \cdot \oint_{C_o} \frac{F(z) K(z, \zeta)\, dz}{[h(z)+\pi i K(z, z)](z-\zeta)} \; .$$

Since $F(z)$ is analytic in D^+, the second integral in (3. 31) can be evaluated by the theory of residues. In this way a form of the solution of (3. 28) can be readily found without a consideration of a Hilbert-Riemann problem.

The second integral in (3. 31) will involve a number of constants c_{kj} equal to the number of zeros (counted with respect to multiplicity) of $h(z) + \pi i K(z, z)$ in D^+. If (3. 31) is actually a solution, the constants can be determined by substituting (3. 31) in (3. 28). The result of doing this is the equation (3. 9) which leads in general to a system of linear algebraic equations to be satisfied by the constants c_{kj}.

If $K(z, z) \neq 0$ for z in $D^+ + C$, and if $h(z) - \pi i K(z, z)$ possesses no zeros in D^+, then (3. 9) is automatically satisfied and the solution of (3. 28) is directly given by (3. 31).

As we have seen, the method of this section can be used to find the solutions of a considerable number of equations which appear in the literature. It is particularly useful for the solution of certain integral equations for which the interval of integration is $[0, 2\pi]$ and the kernel

$K_1(\theta, \omega) = K(e^{i\theta}, e^{i\omega}) = K(z, \zeta)$ is a trigonometric kernel such that $K(z, \zeta)$ is an analytic function in each variable when each of z and ζ is in the closed unit disc.

4. <u>Some Illustrations.</u>

The method described in Section 3 is based on the assumptions that $h(z)$ is analytic when z is in $D^+ + C$, and that $K(z, w)$ is analytic in either variable when each of z, w is in $D^+ + C$. Suppose that instead of $D^+ + C$, the analyticity conditions hold in $D^- + C$. For this exterior case the solution of (3.1) can be found by taking essentially the same steps as those taken above. We see this if we introduce

$$F(w) = \oint_C \frac{K(z, w)\phi(z)\,dz}{z - w}$$

again, and use the limit value as w approaches ζ from the exterior of C, namely

$$F^-(\zeta) = [h(\zeta) - \pi i K(\zeta, \zeta)]\phi(\zeta) + f(\zeta) \quad ,$$

just as we did in the first case.

Some of the other assumptions that have been made can also be relaxed, and our method can be used to solve several other types of functional equations with Cauchy kernels. We proceed to emphasize some of these points by solving a number of illustrative equations.

I. The integral equation

$$(4.1) \qquad \int_0^{2\pi} \frac{\psi(\theta)\,d\theta}{\cos\theta - \cos\omega} = \lambda_1 \psi(\omega) + f_1(\omega)$$

can be transformed into

$$(4.2) \qquad \oint_{C:\,|z|=1} \frac{\zeta\,\phi(z)\,dz}{(z-\zeta)(z\zeta-1)} = i\lambda\,\phi(\zeta) + f(\zeta)$$

by using the substitution $e^{i\theta} = z$ and $e^{i\omega} = \zeta$. Equation (4.2) is an example of

$$\oint_C \frac{K(z,\zeta)\,\phi(z)\,dz}{z-\zeta} = h(\zeta)\,\phi(\zeta) + f(\zeta)$$

in which

$$K(z,w) = \frac{w}{zw-1}$$

is not analytic in $D^+ + C$, but is analytic in each variable if $|z| < 1$ and $|w| < 1$. We suppose that each of $f(\zeta)$ and $\phi(\zeta)$ satisfies a uniform Hölder condition on C and we suppose that (4.2) is to be satisfied for ζ on C excepting $\zeta = \pm 1$. For these values the kernel of (4.2) has a second order pole at $z = \pm 1$ and the integral in (4.2) will in general fail to exist. Notice also that

$$(4.3) \qquad i\lambda\phi\left(\frac{1}{\zeta}\right) + f\left(\frac{1}{\zeta}\right) = \oint_C \frac{\zeta\,\phi(z)\,dz}{(z\zeta-1)(z-\zeta)} = i\lambda\,\phi(\zeta) + f(\zeta) \quad .$$

The limit value of

$$F(w) = (w^2 - 1)w \oint_C \frac{\phi(z)\,dz}{(z-w)(zw-1)}$$

$$= w\left[\oint_C \frac{[\phi(z) + \phi(\frac{1}{z})]\,dz}{z-w} - \oint_C \frac{\phi(z)\,dz}{z}\right]$$

as w approaches ζ from D^+ is

$$F^+(\zeta) = \pi i \zeta[\phi(\zeta) + \phi(\frac{1}{\zeta})] + (\zeta^2 - 1)[i\lambda\,\phi(\zeta) + f(\zeta)] \quad .$$

If we use (4.3), $F^+(\zeta)$ can be written as

$$(4.4) \qquad \frac{F^+(\zeta)}{\lambda(\zeta^2-1) + 2\pi\zeta} = i\phi(\zeta) + \frac{f(\zeta)}{\lambda} - \frac{\pi\zeta}{\lambda}\,\frac{[f(\zeta) + f(\frac{1}{\zeta})]}{[\lambda(\zeta^2-1) + 2\pi\zeta]}$$

or

$$(4.5) \qquad \phi(\zeta) = \frac{F^+(\zeta)}{i[\lambda(\zeta^2-1) + 2\pi\zeta]} - \frac{f(\zeta)}{i\lambda} + \frac{\pi\zeta[f(\zeta) + f(\frac{1}{\zeta})]}{i\lambda[\lambda(\zeta^2-1) + 2\pi\zeta]} \quad .$$

The substitution of this in (4.2) gives

$$(4.6) \qquad \begin{aligned} i\lambda\,\phi(\zeta) + f(\zeta) &= \frac{\zeta}{i}\oint_C \frac{F^+(z)\,dz}{[\lambda(z^2-1) + 2\pi z](z-\zeta)(z\zeta - 1)} \\ &\quad - \frac{\lambda\zeta}{i}\oint_C \frac{(z^2-1)^2 f(z)\,dz}{[\lambda^2(z^2-1)^2 - 4\pi^2 z^2](z-\zeta)(z\zeta-1)} \quad . \end{aligned}$$

Since $F(z)$ is analytic for $|z| < 1$ the first integral on the right of (4.6) can easily be evaluated by the theory of residues. Let us suppose that λ is real

so that we have $\lambda(z^2-1)+2\pi z=\lambda(z-z_1)(z-z_2)$ where $z_1=(-\pi+\sqrt{\lambda^2+\pi^2})/\lambda$ is in the interior of C and $z_2=(-\pi-\sqrt{\lambda^2+\pi^2})/\lambda$ is in the exterior. Then

$$
\begin{aligned}
i\lambda\phi(\zeta)+f(\zeta) &= \frac{\pi\zeta F^+(\zeta)}{[\lambda(\zeta^2-1)+2\pi\zeta](\zeta^2-1)} + \frac{\pi\zeta F^+(\frac{1}{\zeta})\cdot\zeta^2}{[\lambda(\zeta^2-1)+2\pi\zeta](1-\zeta^2)} \\
&\quad + \frac{K_1\zeta}{(\zeta-z_1)(\zeta+z_2)} \\
&\quad - \frac{\lambda\zeta}{i}\oint_C \frac{(z^2-1)^2 f(z)\,dz}{[\lambda^2(z^2-1)^2-4\pi^2z^2](z-\zeta)(z\zeta-1)}\,.
\end{aligned} \tag{4.7}
$$

From (4.4) and (4.3)

$$
\frac{\zeta^2F^+(\frac{1}{\zeta})}{\lambda(1-\zeta^2)+2\pi\zeta} = i\phi(\zeta)+\frac{f(\zeta)}{\lambda}+\frac{\pi\zeta}{\lambda}\,\frac{[f(\zeta)+f(\frac{1}{\zeta})]}{[\lambda(\zeta^2-1)-2\pi\zeta]}\,. \tag{4.8}
$$

Hence, from (4.8) and (4.4)

$$
\begin{aligned}
&\frac{\pi\zeta}{\zeta^2-1}\left[\frac{F^+(\zeta)}{\lambda(\zeta^2-1)+2\pi\zeta}-\frac{F^+(\frac{1}{\zeta})\cdot\zeta^2}{\zeta(1-\zeta^2+2\pi\zeta}\right] \\
&= -\frac{\pi^2\zeta^2}{\lambda(\zeta^2-1)}\left[\frac{1}{\lambda(\zeta^2-1)+2\pi\zeta}+\frac{1}{\lambda(\zeta^2-1)-2\pi\zeta}\right][f(\zeta)+f(\tfrac{1}{\zeta})] \\
&= -\frac{2\pi^2\zeta^2[f(\zeta)+f(\frac{1}{\zeta})]}{\lambda^2(\zeta^2-1)^2-4\pi^2\zeta^2}
\end{aligned}
$$

and we find by substituting this in (4.7) that

$$\phi(\zeta) = -\frac{f(\zeta)}{i\lambda} - \frac{2\pi^2\zeta^2}{i\lambda}\,\frac{[f(\zeta)+f(\frac{1}{\zeta})]}{[\lambda^2(\zeta^2-1)^2 - 4\pi^2\zeta^2]}$$

$$+ \zeta \oint_C \frac{(z^2-1)^2 f(z)\,dz}{[\lambda^2(z^2-1)^2 - 4\pi^2 z^2\zeta](z-\zeta)(z\zeta-1)} + \frac{k\zeta}{(\zeta-z_1)(\zeta+z_2)}$$

is the solution of (4.2) when λ is real.

II. Consider the integrodifferential equation

$$\text{(4.9)} \qquad \frac{1}{\pi i}\oint_C \frac{K(z,\zeta)\phi(z)\,dz}{z-\zeta} = \phi'(\zeta) + f(\zeta) \quad ,$$

where $K(z, w)$ is analytic in each variable when each of z and w lies in $D^+ + C$, and where $K(\zeta,\zeta) \neq 0$. In order to solve this equation, introduce

$$F(w) = \frac{1}{\pi i}\oint_C \frac{K(z,w)\phi(z)\,dz}{z-w}$$

with the limit value

$$F^+(\zeta) = K(\zeta,\zeta)\phi(\zeta) + \frac{1}{\pi i}\oint_C \frac{K(z,\zeta)\phi(z)\,dz}{z-\zeta} \quad .$$

If (4.9) is to hold, this limit value is the same as

$$\text{(4.10)} \qquad F^+(\zeta) = K(\zeta,\zeta)\phi(\zeta) + \phi'(\zeta) + f(\zeta)$$

which can be regarded as a linear differential equation of first order to be solved for $\phi(\zeta)$. If we define $\mu(\zeta)$ to be such that $\mu'(\zeta) = K(\zeta,\zeta)$, the solution of (4.10) is

$$(4.11) \quad \phi(\zeta) = e^{-\mu(\zeta)} \int_{\zeta_o}^{\zeta} e^{\mu(t)}[F^+(t) - f(t)]\,dt + \phi(\zeta_o) \exp[\mu(\zeta_o) - \mu(\zeta)] \quad ,$$

where the path of integration is along C. The substitution of (4.11) in (4.9) yields

$$\phi'(\zeta) + f(\zeta) = -\frac{1}{\pi i} \oint_C \frac{K(z,\zeta)}{z-\zeta} \cdot e^{-\mu(z)} \int_{\zeta_o}^{z} e^{\mu(t)} f(t)\,dt\,dz$$

$$(4.12) \qquad + \frac{1}{\pi i} \oint_C \frac{K(z,\zeta)}{z-\zeta} e^{-\mu(z)} \left[\int_{\zeta_o}^{z} e^{\mu(t)} F^+(t)\,dt + e^{\mu(\zeta_o)} \phi(\zeta_o) \right] dz \quad .$$

Since $\int_{\zeta_o}^{w} e^{\mu(z)} F(z)\,dz$ is an analytic function of w for w in D^+, equation (4.12) is the same as

$$\phi'(\zeta) + f(\zeta) = \mu'(\zeta) e^{-\mu(\zeta)} \left[\int_{\zeta_o}^{\zeta} e^{\mu(t)} F^+(t) + e^{\mu(\zeta_o)} \phi(\zeta_o) \right]$$

$$(4.13) \qquad - \frac{1}{\pi i} \oint_C \frac{K(z,\zeta)}{z-\zeta} e^{-\mu(z)} \int_{\zeta_o}^{z} e^{\mu(t)} f(t)\,dt\,dz \quad .$$

From (4.10) we see that (4.13) reduces to

$$\phi'(\zeta) + f(\zeta) = \mu'(\zeta)e^{-\mu(\zeta)} \left\{ \int_{\zeta_o}^{\zeta} e^{\mu(t)}[\mu'(t)\phi(t) + \phi'(t) + f(t)]\,dt + e^{\mu(\zeta_o)}\phi(\zeta_o) \right\}$$

(4.14)

$$- \frac{1}{\pi i} \oint_C \frac{K(z,\zeta)}{z-\zeta}\, e^{-\mu(z)} \int_{\zeta_o}^{z} e^{\mu(t)} f(t)\,dt\,dz$$

and after an integration by parts (4.14) becomes

$$\phi'(\zeta) - \mu'(\zeta)\phi(\zeta) = -f(\zeta) + \mu'(\zeta)e^{-\mu(\zeta)} \int_{\zeta_o}^{\zeta} e^{\mu(t)} f(t)\,dt$$

(4.15)

$$- \frac{1}{\pi i} \oint_C \frac{K(z,\zeta)}{z-\zeta}\, e^{-\mu(z)} \int_{\zeta_o}^{z} e^{\mu(t)} f(t)\,dt\,dz \quad .$$

The solution of this ordinary differential equation is

$$\phi(\zeta) = -\frac{1}{2} e^{-\mu(\zeta)} \int_{\zeta_o}^{\zeta} e^{\mu(t)} f(t)\,dt - \frac{1}{2} e^{\mu(\zeta)} \int_{\zeta_o}^{\zeta} e^{-\mu(t)} f(t)\,dt$$

(4.16)

$$- \frac{1}{\pi i} e^{\mu(\zeta)} \int_{\zeta_o}^{\zeta} e^{-\mu(\sigma)} \oint_C \frac{K(z,\sigma)}{z-\sigma} \cdot e^{-\mu(z)} \int_{\zeta_o}^{z} e^{\mu(t)} f(t)\,dt\,dz\,d\sigma$$

$$+ c_o e^{\mu(\zeta)} \quad .$$

III. Cur method can be used to solve simultaneous integral equations if the kernels satisfy the analyticity conditions we have imposed above. We illustrate this with an analysis of the simple system

$$
\begin{aligned}
&\phi(\zeta) + \oint_C \frac{K_1(z-\zeta)\phi(z)\,dz}{z-\zeta} + \oint_C \frac{K_2(z-\zeta)\psi(z)\,dz}{z-\zeta} = f(\zeta) , \\
&\psi(\zeta) + \oint_C \frac{K_3(z-\zeta)\phi(z)\,dz}{z-\zeta} + \oint_C \frac{K_4(z-\zeta)\psi(z)\,dz}{z-\zeta} = g(\zeta) ,
\end{aligned} \tag{4.17}
$$

where the functions $K_i(z-w)$ are analytic in each variable when each of z and w is in $D^+ + C$. We suppose that $K_i(0) = \lambda_i \neq 0$. The limit values of

$$
F_1(w) = \oint_C \frac{[K_1(z-w)\phi(z) + K_2(z-w)\psi(z)]}{z-w}\,dz ,
$$

$$
F_2(w) = \oint_C \frac{[K_3(z-w)\phi(z) + K_4(z-w)\psi(z)]}{z-w}\,dz ,
$$

as w approaches on C from D^+, are

$$
\begin{aligned}
F_1^+(\zeta) &= (\pi i\lambda_1 - 1)\phi(\zeta) + \pi i\lambda_2\psi(\zeta) + f(\zeta) , \\
F_2^2(\zeta) &= \pi i\lambda_3\phi(\zeta) + (\pi i\lambda_4 - 1)\psi(\zeta) + g(\zeta) .
\end{aligned} \tag{4.18}
$$

The solution of this system of algebraic equations is

$$
\begin{aligned}
\phi(\zeta) &= \frac{(\pi i\lambda_4 - 1)[F_1^+(\zeta) - f(\zeta)] - \pi i\lambda_2[F_2^+(\zeta) - g(\zeta)]}{\Delta} , \\
\psi(\zeta) &= \frac{(\pi i\lambda_1 - 1)[F_2^+(\zeta) - g(\zeta)] - \pi i\lambda_3[F_1^+(\zeta) - f(\zeta)]}{\Delta} ,
\end{aligned} \tag{4.19}
$$

where $\Delta = (\pi i\lambda_1 - 1)(\pi i\lambda_4 - 1) + \pi^2\lambda_2\lambda_3$. The substitution of (4.19) in (4.17), the subsequent evaluation of the integrals involving the limits, and the use of (4.19) again lead to

$$
\begin{aligned}
(\pi i\lambda_1 + 1)\phi(\zeta) + \pi i\lambda_2\psi(\zeta) &= \frac{\pi i\lambda_2 g(\zeta) - (\pi i\lambda_4 - 1)f(\zeta)}{\Delta} - I_1 - I_2 , \\
\pi i\lambda_3\phi(\zeta) + (\pi i\lambda_4 + 1)\psi(\zeta) &= \frac{\pi i\lambda_3 f(\zeta) - (\pi i\lambda_1 - 1)g(\zeta)}{\Delta} - I_3 - I_4 ,
\end{aligned}
\tag{4.20}
$$

where

$$I_1 = \frac{1}{\Delta}\oint_C \frac{f(z)[\pi i\lambda_3 K_2(z-\zeta) - (\pi i\lambda_4 - 1)K_1(z-\zeta)]\,dz}{z-\zeta} ,$$

$$I_2 = \frac{1}{\Delta}\oint_C \frac{g(z)[\pi i\lambda_2 K_1(z-\zeta) - (\pi i\lambda_1 - 1)K_2(z-\zeta)]\,dz}{z-\zeta} ,$$

$$I_3 = \frac{1}{\Delta}\oint_C \frac{f(z)[\pi i\lambda_3 K_4(z-\zeta) - (\pi i\lambda_4 - 1)K_3(z-\zeta)]\,dz}{z-\zeta} ,$$

$$I_4 = \frac{1}{\Delta}\oint_C \frac{g(z)[\pi i\lambda_2 K_3(z-\zeta) - (\pi i\lambda_1 - 1)K_4(z-\zeta)]\,dz}{z-\zeta} .$$

In this way we see that the solution of the set (4.17) is reduced to the solution of the algebraic equations (4.20).

REFERENCES

[1] Singular Integral Equations by N. I. Muskhelishvili. P. Noordhoff N. V., Groningen, Holland (1953).

[2] Boundary Problems by F. D. Gahov. Gosudarstr. Izdat. Fiz. Mat. Lit., Moscow (1958).

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